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ON THE CALCULATION OF TURBULENT FAN JETS

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The self-similar equations for the dynamic and temperature fields of a forced free fan jet have been numerically integrated within the framework of the standard k–ε model of turbulence. The tables of solutions obtained for the velocity, temperature, and kinetic turbulent energy of the jet, as well as the mean square of the jet-temperature fluctuations at different turbulent Prandtl numbers, are presented. The quantitative parameters of the evolution of the average and turbulent characteristics of the jet flow were determined.

The study of turbulent jets is among the classical problems of aerohydrodynamics. This is explained by the fact that a jet flow is a fluid motion that occurs frequently in nature and technology. Therefore, considerable experimental and theoretical study has been given to such flows. The main result of the efforts of many researches in this field of knowledge is the development of different mathematical models that can be used for solving scientific and engineering problems. Despite the fact that progress has been made towards the direct numerical solution of nonstationary Navier–Stokes equations, the semiempirical simulation based on the phenomenological closure of the equations for the one-point moments of the random hydrodynamic velocity, pressure, temperature, and concentration fields is most widely used. Therefore, in the majority of calculations of different jet flows carried out to date, semiempirical models have been used. However, in many works devoted to these calculations, the problems on the accuracy of the data obtained within the framework of different numerical schemes have received little attention. It seems likely that, for this reason, qualitatively similar data obtained with the use of identical models differ quantitatively. It should be noted that the accuracy of calculating turbulent flows depends on the exactness of the numerical schemes used for this purpose, so that not only the model of turbulence is responsible for the disagreement between the calculation and experimental data. Since a standard is absent, the numerical data obtained are interpreted and the potentialities of the mathematical model used are estimated fairly arbitrarily. Therefore, of great theoretical and practical importance are approaches to the analysis of model equations that made it possible to distinctly differentiate the errors introduced by the numerical procedures from the errors that are due to the weakness of the models used. Such an approach involving the construction of self-similar solutions for different types of jet flows is described in [1–6].

In the present work, the self-similar equations for the scalar temperature field of a free fan jet were numerically integrated with the use of the k–ε model of turbulence.

Formulation of the Problem. A fluid jet flowing from a circular slot (of width r_0 and radius R_0) into the outer space is considered. It is assumed that the physical properties of the fluid in the space and in the jet are identical. Let us assume that the field of the flow is totally turbulent and, therefore, the molecular effects are negligibly small as compared to the turbulence effects. It is also assumed that the fan jet occupies a very narrow region extending along its symmetry axis; therefore the boundary-layer approximations can be used in this case. With these assumptions, the stationary incompressible-fluid flow in the jet being considered can be defined by the following system of partial differential equations:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(\nu_t\frac{\partial u}{\partial y}\right),$$

$$\frac{\partial}{\partial x}(xu)+\frac{\partial}{\partial y}(xv)=0\ , \tag{1}$$

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$$u\frac{\partial \Delta T}{\partial x}+v\frac{\partial \Delta T}{\partial y}=\frac{1}{\mathrm{Pr}}\frac{\partial}{\partial y}\left(\nu_t\frac{\partial \Delta T}{\partial y}\right).$$

In formulas (1), u and v denote the average-velocity components directed, respectively, along the jet axis x and along the normal y to it. In the case where the k–ε–$\langle T'^2\rangle$ model of turbulence is used, the turbulent flow is characterized by the local values of the kinetic turbulent energy k, the rate of its dissipation ε, and the mean square of the temperature fluctuations $\theta = \langle T'^2\rangle$, determined from the following semiempirical transport equations:

$$u\frac{\partial k}{\partial x}+v\frac{\partial k}{\partial y}=\frac{1}{\sigma_k}\frac{\partial}{\partial y}\left(\nu_t\frac{\partial k}{\partial y}\right)+\nu_t\left(\frac{\partial u}{\partial y}\right)^2-\varepsilon\,,$$

$$u\frac{\partial \varepsilon}{\partial x}+v\frac{\partial \varepsilon}{\partial y}=\frac{1}{\sigma_\varepsilon}\frac{\partial}{\partial y}\left(\nu_t\frac{\partial \varepsilon}{\partial y}\right)+c_{\varepsilon 1}\nu_t\frac{\varepsilon}{k}\left(\frac{\partial u}{\partial y}\right)^2-c_{\varepsilon 2}\frac{\varepsilon^2}{k}\,, \tag{2}$$

$$u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=\frac{1}{\sigma_q}\frac{\partial}{\partial y}\left(\nu_t\frac{\partial \theta}{\partial y}\right)+\frac{2}{\mathrm{Pr}}\nu_t\left(\frac{\partial \Delta T}{\partial y}\right)^2-c_{q1}\frac{\varepsilon}{k}\theta\,.$$

The model is closed by the expression

$$\nu_t=c_\mu\frac{k^2}{\varepsilon}\,, \tag{3}$$

relating the coefficient of turbulent viscosity ν_t to the turbulent kinetic energy k and the rate of its dissipation ε with the use of the Kolmogorov–Prandtl relation. The formulation of the problem is completed by the boundary conditions set at the symmetry axis and at the boundary of the jet

$$y=0:\quad v=\frac{\partial u}{\partial y}=\frac{\partial k}{\partial y}=\frac{\partial \varepsilon}{\partial y}=\frac{\partial \Delta T}{\partial y}=\frac{\partial \theta}{\partial y}=0\,,$$

$$y\to y_*:\quad u,k,\varepsilon,\Delta T,\theta\to 0 \tag{4}$$

as well as the initial conditions for the velocity and temperature fields of the free fan jet. However, because only the changes in the main parameters of the asymptotic (self-similar) region of the jet flow are of interest for us, the following integral relations defining the laws of momentum and energy conservation are used instead of the initial conditions:

$$K_0=2\pi x\int_{-y_*}^{y_*}\rho u^2 dy=\mathrm{const}\,,$$

$$Q_0=2\pi x\int_{-y_*}^{y_*}\rho C_p u\Delta T dy=\mathrm{const}\,. \tag{5}$$

The partial differential equations (1) and (2) can be reduced to ordinary differential equations with the use of the similarity conversions

$$u=\left(\frac{K_0}{2\pi\rho\sqrt{c_\mu}}\right)^{1/2} f' x^{-1}\,,\quad \eta=\frac{y}{\sqrt{c_\mu}\,x}\,,\quad v=-\left(\frac{K_0\sqrt{c_\mu}}{2\pi\rho}\right)^{1/2}(f-f'\eta)\,x^{-1}\,,$$

$$k=\frac{K_0}{2\pi\rho\sqrt{c_\mu}}Gx^{-2}\,,\quad \varepsilon=\left(\frac{K_0}{2\pi\rho\sqrt{c_\mu}}\right)^{3/2}Hx^{-4}\,, \tag{6}$$

$$\Delta T = \frac{Q_0}{2\pi\rho C_p}\left(\frac{2\pi\rho}{K_0\sqrt{c_\mu}}\right)^{1/2} hx^{-1}\,,\quad \theta = \frac{Q_0^2}{2\pi\rho C_p^2 K_0\sqrt{c_\mu}}\, qx^{-2}\,.$$

Substitution of (6) into expressions (1)–(5) gives the following equations for the unknown functions $f(\eta)$, $G(\eta)$, $H(\eta)$, $h(\eta)$, and $q(\eta)$ (the prime denotes the derivative with respect to η:

$$\left(\frac{G^2}{H} f''\right)' + ff'' + (f')^2 = 0\,,$$

$$\frac{1}{\sigma_k}\left(\frac{G^2}{H} G'\right)' + fG' + 2f'G + \frac{G^2}{H}(f'')^2 - H = 0\,,$$

$$\frac{1}{\sigma_\varepsilon}\left(\frac{G^2}{H} H'\right)' + fH' + 4f'H + c_{\varepsilon 1} G\,(f'')^2 - c_{\varepsilon 2}\frac{H^2}{G} = 0\,, \tag{7}$$

$$\frac{1}{\Pr}\left(\frac{G^2}{H} h'\right)' + fh' + f'h = 0\,,$$

$$\frac{1}{\sigma_q}\left(\frac{G^2}{H} q'\right)' + fq' + 2f'q - c_{q1}\frac{H}{G} q + \frac{2}{\Pr}\frac{G^2}{H}(h')^2 = 0\,,$$

which should be integrated with account for the conditions

$$f(0) = f''(0) = G'(0) = H'(0) = h'(0) = q'(0) = 0\,;$$

$$\eta \to \eta_*:\ f' \to 0\,,\ \ G \to 0\,,\ \ H \to 0\,,\ \ h \to 0,\ \ q \to 0\,; \tag{8}$$

$$2\int_0^{\eta_*} (f')^2\, d\eta = 1\,,\quad 2\int_0^{\eta_*} f'' h\, d\eta = 1\,.$$

It should be noted that the value of η_* is unknown and should be also determined in the process of solution of the indicated equations, which makes the problem being considered much more complex. An additional trouble of the nonlinear two-point boundary problem (7)–(8) arises at $h \to \eta_*$ when the functions G and H approach zero. To obviate the difficulties of the numerical integration, we will introduce the new function $F(\xi)$ and the variable ξ:

$$f' = F(\xi)\,,\quad \xi = \int_0^{\eta} \frac{H}{G^2}\, d\eta\,. \tag{9}$$

System (7)–(8) can be written, in view of (9), as (the prime denotes the derivative with respect to ξ)

$$f' = \frac{G^2}{H} F\,,\quad F' + fF = 0\,,$$

$$\frac{1}{\sigma_k} G'' + fG' + 2\frac{G^2}{H} FG + (F')^2 - G^2 = 0\,,$$

TABLE 1. Comparison of the Parameters Calculated in [2, 4] with the Data of the Present Work

Parameters	Results [2]	Results [4]	Data of the present work
$f'(0)$	1.45762	1.526	1.457515
$G(0)$	0.33132	—	0.331274
G_{max}	—	0.369	0.336240
$\eta(G_{max})$	—	0.120	0.11837
$H(0)$	1.11387	—	1.113604
H_{max}	—	1.375	1.145930
$\eta(H_{max})$	—	0.120	0.11837
$\eta_{\frac{1}{2}u}$	0.317	0.317	0.31698
η_*	0.65834	—	0.65516
$(\frac{G^2}{H} f'')_{max}$	—	0.340	0.299014
$\eta(\frac{G^2}{H} f'')_{max}$	—	0.240	0.23736
$f(\eta_*)$	0.4723	0.494	0.4722

TABLE 2. Parameters of a Forced Free Turbulent Fan Jet

Pr	$h(0)$	$\eta_{\frac{1}{2}\Delta T}$	$\left(-\frac{G^2}{H} h'\right)_{max}$	$\eta\left(-\frac{G^2}{H} h'\right)_{max}$	$q(0)$		q_{max}		$\eta(q_{max})$	
					c_{q1} = 1.25	c_{q1} = 1.79	c_{q1} = 1.79	c_{q1} = 1.25	c_{q1} = 1.79	c_{q1} = 1.25
0.50	1.270212	0.44285	0.334920	0.30649	0.322816	0.167429	0.371468	0.251768	0.29106	0.36061
0.55	1.289910	0.42423	0.329530	0.29677	0.334818	0.175498	0.378689	0.253150	0.26973	0.33597
0.60	1.309377	0 40735	0.324761	0.28801	0.346505	0.183372	0.386586	0.255640	0.25166	0.31503
0.65	1.328621	0.39216	0.320599	0.27977	0.357907	0.191071	0.394900	0.258954	0.23660	0.29713
0.70	1.347647	0.37846	0.316509	0.27239	0.369055	0.198614	0.403480	0.262859	0.22385	0.28178
0.75	1.366464	0.36595	0.312968	0.26541	0.379974	0.206014	0.412263	0.267118	0.21293	0.26854
0.80	1.385072	0.35447	0.309666	0.25911	0.390685	0.213288	0.421100	0.271667	0.20271	0.25669

TABLE 3. Asymptotic Structure of Free Turbulent Fan Jets

Pr	A_T	S_T	$\frac{\langle v'T'\rangle_{max}}{u_j \Delta T_j}$	$\sqrt{\langle T'^2\rangle_j}/\Delta T_j$		$\sqrt{\langle T'^2\rangle_{max}}/\Delta T_j$	
				c_{q1} = 1.25	c_{q1} = 1.79	c_{q1} = 1.25	c_{q1} = 1.79
0.50	2.319	0.133	0.109	0.447	0.322	0.480	0.395
0.55	2.355	0.127	0.096	0.449	0.325	0.477	0.390
0.60	2.391	0.122	0.085	0.450	0.327	0.475	0.386
0.65	2.426	0.118	0.076	0.450	0.329	0.473	0.383
0.70	2.460	0.114	0.069	0.451	0.331	0.471	0.380
0.75	2.495	0.110	0.063	0.451	0.332	0.470	0.378
0.80	2.529	0.106	0.058	0.451	0.333	0.469	0.376

$$\frac{1}{\sigma_\varepsilon} H'' + fH' + 4\,G^2 F + c_{\varepsilon 1} \frac{H}{G} (F')^2 - c_{\varepsilon 2} GH = 0\,, \tag{10}$$

$$\frac{1}{\sigma_q} q'' + fq' + 2\,\frac{G^2}{H} Fq - c_{q1} Gq + \frac{2}{\mathrm{Pr}} (h')^2 = 0\,,$$

$$\frac{1}{\mathrm{Pr}} h' + fh = 0\,.$$

Note that the use of the new variable ξ allowed us to eliminate the nonlinearity and take the boundary of the jet at infinity, i.e., to assume that $\xi \to \infty$ at $\eta \to \eta_*$. Therefore, the desired functions have the form

$$f(0) = G'(0) = H'(0) = q'(0) = 0\ ; \quad F(\infty) = G(\infty) = H(\infty) = h(\infty) = q(\infty) = 0\ ;$$
$$2\int_0^\infty F^2 \frac{G^2}{H} d\xi = 1\ , \quad 2\int_0^\infty Fh \frac{G^2}{H} d\xi = 1\ . \tag{11}$$

It follows from (9) that

$$\eta_* = \int_0^\infty \frac{G^2}{H} d\xi\ , \tag{12}$$

which gives the explicit dependence for determining the position of the boundary of the turbulent jet flow.

Results of Calculations and Discussion. The ordinary differential equations (10), satisfying conditions (11), were numerally integrated with the use of the Runge–Kutta schemes allowing the automatic choice of the integration step by reducing the boundary problem to the corresponding Cauchy problem. The calculation was begun with $\xi = 0$ and continued as long as $\xi = \xi_\infty$ representing a numerical approximation of the mathematical point $\xi = \infty$. The deficient initial conditions $F(0)$, $G(0)$, $H(0)$, $h(0)$, and $q(0)$ were determined using the so-called shooting method. As the parameters were refined, the quantity ξ_∞ was automatically increased, which excluded its influence on the desired solutions. The calculations were considered as completed when the values of the functions F, G, H, h, and q at "infinity" were equal to approximately 10^{-10} and f approached a constant value at $\xi \to \xi_\infty$. The data of calculations are presented in Tables 1–3. The turbulent Prandtl number changed in the range 0.5–0.8, which corresponds to the experimental data of [7], and the empirical constants of the model $c_\mu = 0.09$, $\sigma_k = 1.0$, $\sigma_\varepsilon = 1.30$, $c_{\varepsilon 1} = 1.44$, $c_{\varepsilon 2} = 1.92$, $\sigma_q = 0.6923$, and $c_{q1} = 1.25$ (1.79) were taken to be equal to the standard values. It follows from formulas (6) that, for uniform profiles of the velocity u_0 and temperature ΔT_0 the following relations are fulfilled at the cut of the slot:

$$\frac{u_j}{u_0} = 2.661 \left(\frac{R_0}{r_0}\right)^{1/2} \left(\frac{x}{r_0}\right)^{-1}, \quad \frac{k_j}{u_0^2} = 1.104 \left(\frac{R_0}{r_0}\right)\left(\frac{x}{r_0}\right)^{-2}, \tag{13}$$

$$\frac{\varepsilon_j r_0}{u_0^3} = 6.777 \left(\frac{R_0}{r_0}\right)^{3/2} \left(\frac{x}{r_0}\right)^{-4}, \quad \frac{\Delta T_j}{\Delta T_0} = \frac{h(0)}{(c_\mu)^{1/4}} \left(\frac{R_0}{r_0}\right)^{1/2} \left(\frac{x}{r_0}\right)^{-1} = A_T \left(\frac{R_0}{r_0}\right)^{1/2} \left(\frac{x}{r_0}\right)^{-1}.$$

The jet is expanded linearly in the self-similar regime. The calculation gave the following values for the expansion angles:

$$\frac{y_{\frac{1}{2}u}}{x} = 0.095\ , \quad \frac{y_{\frac{1}{2}\Delta T}}{x} = \sqrt{c_\mu}\, \eta_{\frac{1}{2}\Delta T} = S_T\ .$$

The profiles $G(\eta)$ and $H(\eta)$ of the turbulent characteristics k and ε have a small dip near the jet axis. This is evidence of the physical fact that the reproduction of the turbulent energy at the axis of the flow is maximum. As for the distribution of the turbulent-transfer flows $\langle u'v' \rangle$ and $\langle v'T' \rangle$

$$\frac{\langle u'v' \rangle}{u_j^2} = -\frac{\sqrt{c_\mu}}{(f'(0))^2} \frac{G^2}{H} f''\ , \quad \frac{\langle v'T' \rangle}{u_j \Delta T_j} = -\frac{\sqrt{c_\mu}}{\mathrm{Pr}\, f'(0)\, h(0)} \frac{G^2}{H} h'\ ,$$

they have a form usual for free jet flows. The maximum value of $\langle u'v' \rangle / u_j^2$ is equal to 0.042 and is reached at $\eta / \eta_{\frac{1}{2}u} = 0.757$.

The correlation $\langle v'T'\rangle$ responsible for the transverse heat transfer does not changes its sign. Consequently, the transverse-velocity fluctuations cause a heat transfer in the direction from the center of the jet to its boundaries, i.e., from the fluid region with a higher temperature to the region of the fluid heated to a lower temperature. The value of $\langle v'T'\rangle_{max}/(u_j\Delta T_j)$ is larger than the value of $\langle u'v'\rangle_{max}/u_j^2$. This points to the fact that the mechanisms of momentum transfer and heat transfer in turbulent jets are not identical. An increase in the turbulent Prandtl number leads to a decrease in the thickness of the boundary layer with heat transfer, with the result that the region occupied by the jet flow is heated to a lower temperature and the temperature fluctuations decrease. However, it should be noted that the dependence of the quantity $\sqrt{\langle T'^2\rangle}/\Delta T_j$ on Pr is nonlinear in character: the region of maximum intensity shifts to the axis of the jet; in this case, the dips of the profiles $q(\eta)$ decrease. We also investigated the influence of the empirical constant c_{q1} on the final results. The model equations were numerically solved for the values of c_{q1} = 1.25 (1.79) used at present in scientific and engineering calculations of the turbulent stresses and the heat transfer in incompressible-fluid jets. It was found that the quantities $\sqrt{\langle T'^2\rangle_j}/\Delta T_j$ and $\sqrt{\langle T'^2\rangle_{max}}/\Delta T_j$ at c_{q1} = 1.79 are smaller than those at c_{q1} = 1.25 by approximately 27% and 19% respectively. It should be noted that the self-similar profiles $f(\eta)$, $G(\eta)$, $h(\eta)$, $H(\eta)$, and $q(\eta)$ are independent of the empirical constant c_μ. This means that it is a scale factor of the model. The results of our calculations also point to the fact that, for a free fan jet, there exists a line $y_* = 0.197x$, separating the turbulent zone from the stationary environment; in this case, $y_*/y_{\frac{1}{2}u} = 2.067$ and $v_*/u_j = -0.0972$.

NOTATION

C_p, specific heat capacity at a constant pressure, J/(kg·K); K_0, kinetic momentum of a jet, kg·m/sec^2; k, kinetic turbulent energy, m^2/sec^2; Pr, turbulent Prandtl number; Q_0, excessive enthalpy flow, J/sec; T and T', average and fluctuation temperatures, K; $\langle T'^2\rangle$, mean square of the temperature fluctuations, K^2; u and v, longitudinal and transverse components of the average velocity, m/sec; u', v', fluctuational velocity components, m/sec; $\langle u'v'\rangle$, turbulent shear stress, m^2/sec^2; $\langle v'T'\rangle$, turbulent heat flow, m·K/sec; x and y, longitudinal and transverse coordinates, m; $y_{\frac{1}{2}u}$, $y_{\frac{1}{2}\Delta T}$, half-width of the jet at which $u = u_{max}/2$ and $\Delta T = \Delta T_{max}/2$, m; $\Delta T = T - T_\infty$, excessive temperature, K; ε, rate of dissipation of the kinetic turbulent energy, m^3/sec^3; ν_t, coefficient of turbulent viscosity, m^2/sec; ρ, density of the fluid, kg/m^3. Subscripts: ∞, surrounding fluid; max, maximum value; j, axial line of the jet; 0, at the output of the jet; $*$, at the boundary of the jet; t, turbulent.

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